

Automorphisms and Verma modules for Generalized Schrödinger-Virasoro algebras *

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Abstract

Let \mathbb{F} be a field of characteristic 0, G an additive subgroup of \mathbb{F} , $\alpha \in \mathbb{F}$ satisfying $\alpha \notin G, 2\alpha \in G$. We define a class of infinite-dimensional Lie algebras which are called generalized Schrödinger-Virasoro algebras and use $\mathfrak{gsv}[G, \alpha]$ to denote the one corresponding to G and α . In this paper the automorphism group and irreducibility of Verma modules for $\mathfrak{gsv}[G, \alpha]$ are completely determined.

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1. Introduction

The Schrödinger-Virasoro algebra \mathfrak{sv} is defined to be a Lie algebra with \mathbb{F} -basis $\{L_n, M_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$ subject to the following Lie brackets:

$$[L_m, L_n] = (n - m)L_{n+m}, [L_m, M_n] = nM_{n+m}, [L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}},$$

$$[M_m, M_n] = 0, [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (n - m)M_{m+n+1}, [M_m, Y_{n+\frac{1}{2}}] = 0.$$

It is easy to see that \mathfrak{sv} is a semi-direct product of the Witt algebra $\mathfrak{Vir}_0 = \bigoplus \mathbb{C}L_n$ and the two-step nilpotent infinite-dimensional Lie algebra $\mathfrak{h} = \bigoplus \mathbb{C}M_n \bigoplus \mathbb{C}Y_{n+\frac{1}{2}}$. This infinite-dimensional Lie algebra was originally introduced in [3] by looking at the invariance of the free Schrödinger equation in (1+1)dimensions $(2\mathcal{M}\partial_t - \partial_r^2)\psi =$

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0. The structure and representation theory of \mathfrak{sv} have been studied by C. Roger and J. Unterberger in [11]. The irreducible weight modules with finite-dimensional weight spaces over \mathfrak{sv} are classified in [7].

In order to investigate vertex representations of \mathfrak{sv} , J. Unterberger (see [14]) introduced a class of infinite-dimensional Lie algebras $\tilde{\mathfrak{sv}}$ called the extended Schrödinger-Virasoro Lie algebra, which can be viewed as an extension of \mathfrak{sv} by a conformal current with conformal weight 1. The extended Schrödinger-Virasoro Lie algebra $\tilde{\mathfrak{sv}}$ is a vector space spanned by a basis $\{L_n, M_n, N_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$ with the following commutation relations:

$$\begin{aligned} [L_m, L_n] &= (n - m)L_{n+m}, [M_m, M_n] = 0, [N_m, N_n] = 0, [N_m, M_n] = 2M_{m+n}, \\ [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] &= (n - m)M_{m+n+1}, [L_m, M_n] = nM_{n+m}, [L_m, N_n] = nN_{n+m}, \\ [L_m, Y_{n+\frac{1}{2}}] &= (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}}, [N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}}, [M_m, Y_{n+\frac{1}{2}}] = 0. \end{aligned}$$

For all $m, n \in \mathbb{Z}$. The structure of $\tilde{\mathfrak{sv}}$ has been studied in [2].

Recently, a number of new classes of infinite-dimensional Lie algebras over a field of characteristic 0 were discovered by several authors (see [6] [8] [12] [13]). Among those algebras, are the generalized Witt algebras, the generalized Virasoro algebras, the Lie algebras of generalized Weyl type and the generalized Heisenberg-Virasoro algebra.

Let \mathbb{F} be a field of characteristic 0, G an additive subgroup of \mathbb{F} , $\alpha \in \mathbb{F}$ such that $\alpha \notin G, 2\alpha \in G$. Motivated by the above algebras, we introduce a new class of Lie algebras which are called the generalized Schrödinger-Virasoro Lie algebras which include the Schrödinger-Virasoro Lie algebra \mathfrak{sv} as a special case. We use $\mathfrak{gsv}[G, \alpha]$ to denote the one corresponding to G and α . In this paper we mainly study the automorphism group $\text{Aut}(\mathfrak{gsv}[G, \alpha])$ and the irreducibility of Verma modules over the generalized Schrödinger-Virasoro Lie algebra $\mathfrak{gsv}[G, \alpha]$.

The paper is organized as follows. In section 2, we introduce the generalized Schrödinger-Virasoro Lie algebra $\mathfrak{gsv}[G, \alpha]$. The necessary and sufficient conditions of isomorphism between two of these algebras are determined. In section 3, we determine the automorphism group of $\mathfrak{gsv}[G, \alpha]$. In section 4, a Verma module $V(c, h)$ over the generalized Schrödinger-Virasoro Lie algebra $\mathfrak{gsv}[G, \alpha]$ is defined and its irreducibility is completely determined.

Throughout the article, we denote \mathbb{Z} the set of integers, \mathbb{N} the set of non-negative integers.

2. Generalized Schrödinger-Virasoro algebras

Let \mathbb{F} be a field of characteristic 0, G an additive proper subgroup of \mathbb{F} , $\alpha \in \mathbb{F}$ satisfying $\alpha \notin G$ while $2\alpha \in G$. We set $G_1 = \alpha + G$ and $T = G \cup G_1$. It is obvious that T is an additive subgroup of \mathbb{F} . In this section we want to make a natural

generalization of Schrödinger-Virasoro algebra \mathfrak{sv} , this leads us to the following definition.

Definition 2.1. The generalized Schrödinger-Virasoro algebra $\mathfrak{gsv}[G, \alpha]$ is defined to be the Lie algebra with \mathbb{F} basis $\{L_u, M_u, Y_{u+\alpha} \mid u \in G\}$ subject to the following Lie brackets:

$$[L_u, L_v] = (v - u)L_{u+v}, [L_u, M_v] = vM_{u+v}, [L_u, Y_{v+\alpha}] = (v + \alpha - \frac{u}{2})Y_{u+v+\alpha},$$

$$[M_u, M_v] = 0, [Y_{u+\alpha}, Y_{v+\alpha}] = (v - u)M_{u+v+2\alpha}, [M_u, Y_{v+\alpha}] = 0.$$

It is straightforward to see that $\mathfrak{gsv}[G, \alpha]$ is T -graded:

$$\mathfrak{gsv}[G, \alpha] = \bigoplus_{x \in T} \mathfrak{gsv}[G, \alpha]_x,$$

where

$$\mathfrak{gsv}[G, \alpha]_x = \begin{cases} \mathbb{F}L_x \oplus \mathbb{F}M_x, & x \in G, \\ \mathbb{F}Y_x, & x \in G_1. \end{cases}$$

The homogenous spaces are the root spaces according to the Cartan subalgebra $\mathfrak{gsv}[G, \alpha]_0$.

One can see that if $G = \mathbb{Z}$ and $\alpha = \frac{1}{2}$, then the generalized Schrödinger Virasoro algebra $\mathfrak{gsv}[G, \alpha]$ is nothing but the Schrödinger Virasoro algebra \mathfrak{sv} defined by Henkle in [3].

Denote $L = \bigoplus_{u \in G} \mathbb{F}L_u, M = \bigoplus_{u \in G} \mathbb{F}M_u, Y = \bigoplus_{v \in G} \mathbb{F}Y_{\alpha+v}, I = M \oplus Y$. Obviously, L is the centerless generalized Virasoro algebra (see [12]). M and I are ideals of L .

Lemma 2.2. I is the unique maximal ideal of $\mathfrak{gsv}[G, \alpha]$.

Proof. It is obvious that I is an ideal of $\mathfrak{gsv}[G, \alpha]$. Moreover, I is a maximal ideal of $\mathfrak{gsv}[G, \alpha]$ since $\mathfrak{gsv}[G, \alpha]/I$ is a simple generalized Witt algebra.

Now suppose I_1 is another maximal ideal of $\mathfrak{gsv}[G, \alpha]$, we need to prove $I_1 = I$. $\forall z \in I_1, z \neq 0$, suppose $z = l + x + y$, where $l \in L, x \in M, y \in Y$. Assume $l \neq 0$. It is obvious that $0 \neq adM_v(z) \in M \cap I_1$ for any $0 \neq v \in G$ by using the Lie bracket of $\mathfrak{gsv}[G, \alpha]$. Suppose

$$z_0 = adM_v(z) = a_1M_{u_1} + a_2M_{u_2} + \cdots + a_nM_{u_n}.$$

It is well known that the submodules of weight module are also weight modules, thus all homogeneous components of z_0 are contained in I_1 , hence we can find some element $0 \neq u \in G$ such that $M_u \in I_1$. Thus $M \subseteq I_1$ due to the fact $[L_v, M_u] = uM_{u+v} \in I_1$ for any $v \in G$. Then

$$0 \neq z' = l + y \in I_1.$$

If $y = 0$, $z' = l \in I_1$, this implies that $I_1 = \mathfrak{gs}\mathfrak{v}[G, \alpha]$, which is a contradiction. Suppose $y \neq 0$. Since $[Y_{\alpha+v}, l + y] \subseteq (M \oplus Y) \cap I_1$ for any $v \in G$ and $M \subseteq I_1$, we claim that there exists some $0 \neq y' \in I_1 \cap Y$. Then all homogeneous components of y' are contained in I_1 . This implies that $Y \subset I_1, l \in I_1$, and therefore $L \subset I_1$. Therefore $I_1 = \mathfrak{gs}\mathfrak{v}[G, \alpha]$, which is a contradiction. Thus $l = 0, z = m + y$. So $I_1 \subseteq M \oplus Y = I$. By the maximality of I and I_1 , we have $I_1 = I$. \square

Let G' be another additive proper subgroup of \mathbb{F} , $\alpha' \in \mathbb{F}$, such that $\alpha' \notin G', 2\alpha' \in G'$. Denote $G'_1 = \alpha' + G', T' = G' \cup G'_1$.

Correspond to G and G' , there are two generalized Virasoro algebras: $Vir[G]$ and $Vir[G']$. About $Vir[G]$ and $Vir[G']$, one fact was pointed out in [12] that the following Lemma can be obtained by using Theorem 4.2 in [1]. However, it can be proved straightforward.

Lemma 2.3.^[12] $Vir[G] \simeq Vir[G']$ if and only if there exists $a \in \mathbb{F}^*$ such that $aG = G'$.

Proof. Since $Vir[G] \simeq Vir[G'] \Leftrightarrow Vir[G]/C \simeq Vir[G']/C'$, where C and C' are the center of $Vir[G]$ and $Vir[G']$ respectively, we view $Vir[G]$ and $Vir[G']$ as the generalized Witt algebras. Let $\theta : Vir[G] \rightarrow Vir[G']$ be an isomorphism of Lie algebras. Since $\mathbb{F}L_0$ and $\mathbb{F}L'_0$ are the unique Cartan subalgebras of $Vir[G]$ and $Vir[G']$ respectively, there exists $a \in \mathbb{F}^*$ such that $\theta(L_0) = aL'_0$. By using θ to $[L_0, L_x] = xL_x$, we have $x\theta(L_x) = [\theta(L_0), \theta(L_x)] = a[L'_0, \theta(L_x)]$, so

$$[L'_0, \theta(L_x)] = a^{-1}x\theta(L_x).$$

Thus $a^{-1}x \in G'$ since G' is the weight set according to the unique Cartan algebra $\mathbb{F}L'_0$. Hence $a^{-1}G \subseteq G'$. Similarly, by using θ^{-1} to $[L'_0, L'_x]$, we get $aG' \subseteq G$. So, $aG' = G$.

The sufficiency is obvious. This completes the proof of Lemma 2.3.

Theorem 2.4. $\mathfrak{gs}\mathfrak{v}[G, \alpha]$ and $\mathfrak{gs}\mathfrak{v}[G', \alpha']$ are isomorphic if and only if there exists $a \in \mathbb{F}^*$ such that $G' = aG, T' = aT$.

Proof. Let $\theta : \mathfrak{gs}\mathfrak{v}[G, \alpha] \rightarrow \mathfrak{gs}\mathfrak{v}[G', \alpha']$ be an isomorphism of Lie algebras, I, I' be the maximal ideals of $\mathfrak{gs}\mathfrak{v}[G, \alpha]$ and $\mathfrak{gs}\mathfrak{v}[G', \alpha']$ respectively. By Lemma 2.2, we have $\theta(I) = I'$, thus there is an isomorphism of the generalized Witt algebras induced by θ :

$$\bar{\theta} : \mathfrak{gs}\mathfrak{v}[G, \alpha]/I \rightarrow \mathfrak{gs}\mathfrak{v}[G', \alpha']/I'.$$

By applying Lemma 2.3, there exists $a \in \mathbb{F}^*$ such that

$$G' = aG, \tag{1}$$

and $\bar{\theta}(\bar{L}_u) = \chi(u)a^{-1}\bar{L}'_{au}$, for some $\chi \in Hom(G, \mathbb{F}^*)$. So we can assume

$$\theta(L_u) = \chi(u)a^{-1}L'_{au} + m'_u{}^L + y'_u{}^L. \tag{2}$$

Suppose $\theta(Y_{\alpha+v}) = \sum_{i=1}^s c_{v'_i} Y'_{\alpha'+v'_i} + m_v'^Y$. By using θ to both sides of the identity $(\alpha + v)Y_{\alpha+v} = [L_0, Y_{\alpha+v}]$, we have

$$\begin{aligned}
(\alpha + v)\theta(Y_{\alpha+v}) &= [\theta(L_0), \theta(Y_{\alpha+v})] \\
&= [a^{-1}L'_0 + m_0'^L + y_0'^L, \sum_{i=1}^s c_{v'_i} Y'_{\alpha'+v'_i} + m_v'^Y] \\
&= [a^{-1}L'_0 + y_0'^L, \sum_{i=1}^s c_{v'_i} Y'_{\alpha'+v'_i} + m_v'^Y] \\
&= a^{-1} \sum_{i=1}^s c_{v'_i} [L'_0, Y'_{\alpha'+v'_i}] + m' \\
&= a^{-1} \sum_{i=1}^s c_{v'_i} (\alpha' + v'_i) Y'_{\alpha'+v'_i} + m'
\end{aligned}$$

for some $m' \in M'$. Then we have

$$a^{-1} \sum_{i=1}^s c_{v'_i} (\alpha' + v'_i) Y'_{\alpha'+v'_i} + m' = (\alpha + v) \left(\sum_{i=1}^s c_{v'_i} Y'_{\alpha'+v'_i} + m_v'^Y \right).$$

By comparing the coefficients, we get

$$v'_i = -\alpha' + a(\alpha + v), \quad \forall i \in \{1, \dots, s\}.$$

Hence $s = 1$ and

$$\theta(Y_{\alpha+v}) = c'_v Y'_{a(\alpha+v)} + m_v'^Y, \quad (3)$$

where $c'_v = c_{-\alpha' + a(\alpha+v)}$.

As we know that $\theta(I) = I'$, M and M' are the centers of I and I' respectively, so there exists an induced isomorphism

$$\begin{aligned}
\bar{\theta}: \bar{Y} = I/M &\rightarrow I'/M' = \bar{Y}' : \\
\bar{Y}_{\alpha+v} &\mapsto c'_v \bar{Y}'_{a(\alpha+v)}.
\end{aligned}$$

Thus we have the following isomorphisms of vector spaces:

$$Y \xrightarrow{\pi} \bar{Y} \xrightarrow{\bar{\theta}} \bar{Y}' \xrightarrow{\pi'^{-1}} Y'$$

where π and π' are the canonical homomorphisms of vector spaces. Thus

$$\begin{aligned}
\pi'^{-1} \bar{\theta} \pi: \bigoplus_{v \in G} Y_{\alpha+v} = Y &\rightarrow Y' = \bigoplus_{v' \in G'} Y_{\alpha'+v'} : \\
Y_{\alpha+v} &\mapsto c'_v Y'_{a(\alpha+v)}
\end{aligned}$$

is an isomorphism of vector spaces. Thus $\alpha' + G' = a(\alpha + G)$. This and (1) give us

$$G' = aG, T' = aT.$$

On the other hand, if $G' = aG, T' = aT$, we define a map

$$\theta : \mathfrak{gs}\mathfrak{v}[G, \alpha] \rightarrow \mathfrak{gs}\mathfrak{v}[G', \alpha'] : L_u \mapsto a^{-1}L'_{au}, M_u \mapsto a^{-1}M'_{au}, Y_{\alpha+u} \mapsto a^{-1}Y'_{a(\alpha+u)}.$$

It is straightforward to check that θ is an isomorphism of Lie algebras. This completes the proof of Theorem 2.4. \square

Corollary 2.5. The map:

$$\begin{aligned} \theta : \mathfrak{sv} = \mathfrak{gs}\mathfrak{v}[\mathbb{Z}, \tfrac{1}{2}] &\rightarrow \mathfrak{gs}\mathfrak{v}[G, \alpha] : \\ L_i &\mapsto (2\alpha)^{-1}L_{2\alpha i}, \\ M_i &\mapsto (2\alpha)^{-1}M_{2\alpha i}, \\ Y_{i+\frac{1}{2}} &\mapsto (2\alpha)^{-1}Y_{2\alpha i+\alpha} \end{aligned}$$

extends uniquely to a Lie algebra isomorphism between $\mathfrak{sv} = \mathfrak{gs}\mathfrak{v}[\mathbb{Z}, \frac{1}{2}]$ and $\mathfrak{gs}\mathfrak{v}[2\mathbb{Z}\alpha, \alpha]$.

Lemma 2.6. Let G, α, G', α' be as in Theorem 2.4, $\theta : \mathfrak{gs}\mathfrak{v}[G, \alpha] \rightarrow \mathfrak{gs}\mathfrak{v}[G', \alpha']$ be a Lie algebra isomorphism. Then

$$\begin{cases} \theta(L_u) = \chi(u)a^{-1}L'_{au} + m'_u{}^L + y'_u{}^L, \\ \theta(M_u) = b_u M'_{au}, \\ \theta(Y_{\alpha+v}) = c_v Y'_{a(\alpha+v)} + m'_v{}^Y. \end{cases}$$

for some $b_u, c_v \in \mathbb{F}^*$, $m'_v{}^Y, m'_u{}^L \in M'$, $y'_u{}^L \in Y'$ and $\chi \in \text{Hom}(G, \mathbb{F}^*)$, where $a \in \mathbb{F}^*$ such that $aG = G', aT = T'$.

Proof. The first and the third identities are given in the proof of Theorem 2.4 (i.e., identities (2) and (3)), we only need to prove the second formula. Since $\mathbb{F}M_0$ (resp. $\mathbb{F}M'_0$) is the center of $\mathfrak{gs}\mathfrak{v}[G, \alpha]$ (resp. $\mathfrak{gs}\mathfrak{v}[G', \alpha']$), we have

$$\theta(M_0) = b_0 M'_0.$$

for some $b_0 \in \mathbb{F}^*$. Recall that $\theta(I) = I'$, $\theta(M) = M'$, for $u \neq 0$, suppose

$$\theta(M_u) = \sum_{i=1}^n b_{u_i} M'_{u_i} + b'_0 M'_0,$$

where $u_i \neq 0, b_{u_i} \neq 0$. Then we have

$$\begin{aligned}
u\theta(M_u) &= \theta([L_0, M_u]) \\
&= [a^{-1}L'_0 + m_0'^L + y_0'^L, \sum_{i=1}^n b_{u_i}'M_{u_i}' + b_0'M_0'] \\
&= [a^{-1}L'_0, \sum_{i=1}^n b_{u_i}'M_{u_i}'] = a^{-1} \sum_{i=1}^n b_{u_i}'[L'_0, M_{u_i}'] \\
&= a^{-1} \sum_{i=1}^n b_{u_i}'u_i'M_{u_i}'.
\end{aligned}$$

So we have

$$u(\sum_{i=1}^n b_{u_i}'M_{u_i}' + b_0'M_0') = a^{-1} \sum_{i=1}^n b_{u_i}'u_i'M_{u_i}'.$$

Thus $b_0' = 0$, $\sum_{i=1}^n (ub_{u_i}' - a^{-1}b_{u_i}'u_i')M_{u_i}' = 0$. Hence $u_i' = au$, $\forall i \in \{1, 2, \dots, n\}$, and $n = 1$. This gives

$$\theta(M_u) = b_u M_{au}', \quad \forall u \in G.$$

□

3. Automorphism group of $\mathfrak{gs}\mathfrak{v}[G, \alpha]$

In this section, we first construct three kinds of automorphisms of $\mathfrak{gs}\mathfrak{v}[G, \alpha]$ which are not inner, then determine the automorphism group of $\mathfrak{gs}\mathfrak{v}[G, \alpha]$ completely. Throughout this section we always assume that \mathbb{F} is an algebraic closed field with characteristic 0.

Lemma 3.1. (i) For any $\chi \in \text{Hom}(T, \mathbb{F}^*)$ and $b \in \mathbb{F}^*$, the map

$$\begin{aligned}
\sigma_b^\chi : \mathfrak{gs}\mathfrak{v}[G, \alpha] &\rightarrow \mathfrak{gs}\mathfrak{v}[G, \alpha] : \\
L_u &\mapsto \chi(u)L_u, \\
M_u &\mapsto b\chi(u)M_u, \\
Y_{\alpha+u} &\mapsto b^{\frac{1}{2}}\chi(\alpha+u)Y_{\alpha+u}
\end{aligned}$$

is an automorphism of $\mathfrak{gs}\mathfrak{v}[G, \alpha]$. Furthermore, the set $\{\sigma_b^\chi | \chi \in \text{Hom}(T, \mathbb{F}^*), b \in \mathbb{F}^*\}$ forms a subgroup of $\text{Aut}(\mathfrak{gs}\mathfrak{v}[G, \alpha])$ and this subgroup is isomorphic to $(\text{Hom}(T, \mathbb{F}^*) \times \mathbb{F}^*)$, where $\sigma_{b_1}^{\chi_1} \sigma_{b_2}^{\chi_2} = \sigma_{b_1 b_2}^{\chi_1 \chi_2}$ for $b_1, b_2 \in \mathbb{F}^*$ and $\chi_1, \chi_2 \in \text{Hom}(T, \mathbb{F}^*)$.

(ii) For any $a \in S(G, T) := \{a \in \mathbb{F}^* | aG = G, aT = T\}$, the map

$$\begin{aligned}
\varphi_a : \mathfrak{gs}\mathfrak{v}[G, \alpha] &\rightarrow \mathfrak{gs}\mathfrak{v}[G, \alpha] : \\
L_u &\mapsto a^{-1}L_{au}, \\
M_u &\mapsto a^{-1}M_{au}, \\
Y_{\alpha+u} &\mapsto a^{-1}Y_{a(\alpha+u)}
\end{aligned}$$

is an automorphism of $\mathfrak{gsv}[G, \alpha]$, and $\{\varphi_a | a \in S(G, T)\}$ forms a subgroup of $Aut(\mathfrak{gsv}[G, \alpha])$, where $\varphi_a \varphi_b = \varphi_{ab}$ for $a, b \in \mathbb{F}^*$.

(iii) Denote

$$\mathcal{A} := \{\underline{a} = (a_u)_{u \in G} | (v - u)a_{u+v} = va_v - ua_u, a_{-u} = -a_u, \forall u, v \in G\}.$$

The map $\phi_{\underline{a}} : \mathfrak{gsv}[G, \alpha] \rightarrow \mathfrak{gsv}[G, \alpha]$:

$$L_u \mapsto L_u + a_u M_u, \quad M_u \mapsto M_u, \quad Y_{\alpha+v} \mapsto Y_{\alpha+v}$$

is an automorphism. $\Phi = \{\phi_{\underline{a}} | \underline{a} \in \mathcal{A}\}$ forms a subgroup of $Aut(\mathfrak{gsv}[G, \alpha])$, where $\phi_{\underline{a}} \phi_{\underline{b}} = \phi_{\underline{a+b}}$ for $\underline{a}, \underline{b} \in \mathcal{A}$.

Proof. The proof is straightforward, we omit the details. \square

Theorem 3.2. Let $Inn(\mathfrak{gsv}[G, \alpha])$ be the inner automorphism subgroup of $Aut(\mathfrak{gsv}[G, \alpha])$. Then we have

$$Aut(\mathfrak{gsv}[G, \alpha]) \simeq (((Hom(T, \mathbb{F}^*) \times \mathbb{F}^*) \rtimes S(G, T)) \ltimes \Phi) \ltimes Inn(\mathfrak{gsv}[G, \alpha]).$$

Proof. For any $\theta \in Aut(\mathfrak{gsv}[G, \alpha])$, by Lemma 2.6, we can assume

$$\begin{aligned} \theta(L_u) &= \chi_1(u)a^{-1}L_{au} + m_u^L + y_u^L, \\ \theta(M_u) &= \chi'_1(u)a^{-1}M_{au}, \\ \theta(Y_{\alpha+v}) &= \chi'_1(\alpha+v)a^{-1}Y_{a(\alpha+v)} + m_v^Y, \end{aligned}$$

where $\chi_1 \in Hom(G, \mathbb{F}^*)$, $\chi'_1 : T \rightarrow \mathbb{F}^*$ is a map, $a \in S(G, T)$, $m_u^L, m_v^Y \in M$, $y_u^L \in Y$.

By applying θ to both sides of the identities:

$$[L_{-u}, M_u] = uM_0,$$

$$[L_u, Y_{\alpha+v}] = (\alpha + v - \frac{u}{2})Y_{\alpha+v+u},$$

and

$$[Y_{\alpha+u}, Y_{\alpha+v}] = (v - u)M_{u+v+2\alpha},$$

we obtain

$$\chi'_1(u) = b\chi_1(u), \quad \forall u \in G, \tag{4}$$

where $b = \chi'_1(0) \in \mathbb{F}^*$,

$$\chi_1(u)\chi'_1(\alpha+v) = \chi'_1(u+v+\alpha), \tag{5}$$

and

$$\chi'_1(\alpha+u)\chi'_1(\alpha+v) = \chi'_1(u+v+2\alpha). \tag{6}$$

for $u, v \in G$.

Write

$$\chi(x) = \begin{cases} \chi_1(x), & \text{if } x \in G, \\ b^{-\frac{1}{2}}\chi'(x), & \text{if } x \in G_1 = \alpha + G, \end{cases}$$

and by using (4), (5) and (6), one can easily see that $\chi \in \text{Hom}(T, \mathbb{F}^*)$. By Lemma 3.1 (i),

$$\begin{aligned} \sigma_b^\chi : \mathfrak{gsv}[G, \alpha] &\rightarrow \mathfrak{gsv}[G, \alpha] : \\ L_u &\mapsto \chi(u)L_u, \\ M_u &\mapsto b\chi(u)M_u, \\ Y_{\alpha+v} &\mapsto b^{\frac{1}{2}}\chi(\alpha+v)Y_{\alpha+v} \end{aligned}$$

is an automorphism of $\mathfrak{gsv}[G, \alpha]$.

For $(\sigma_b^\chi)^{-1}\theta \in \text{Aut}(\mathfrak{gsv}[G, \alpha])$, it is obvious that

$$\begin{aligned} (\sigma_b^\chi)^{-1}\theta : \mathfrak{gsv}[G, \alpha] &\rightarrow \mathfrak{gsv}[G, \alpha] : \\ L_u &\mapsto a^{-1}L_{au} + m_{1u}^L + y_{1u}^L, \\ M_u &\mapsto a^{-1}M_{au}, \\ Y_{\alpha+v} &\mapsto a^{-1}Y_{a(\alpha+v)} + m_{1v}^Y, \end{aligned}$$

where $m_{1u}^L, m_{1v}^Y \in M$, $y_{1u}^L \in Y$.

Set

$$\begin{aligned} \varphi_a : \mathfrak{gsv}[G, \alpha] &\rightarrow \mathfrak{gsv}[G, \alpha] : \\ L_u &\mapsto a^{-1}L_{au}, \\ M_u &\mapsto a^{-1}M_{au}, \\ Y_{\alpha+v} &\mapsto a^{-1}Y_{a(\alpha+v)}. \end{aligned}$$

By Lemma 3.1 (ii), $\varphi_a \in \text{Aut}(\mathfrak{gsv}[G, \alpha])$. Set $\tau = (\varphi_a)^{-1}(\sigma_b^\chi)^{-1}\theta$, then $\tau \in \text{Aut}(\mathfrak{gsv}[G, \alpha])$. More precisely

$$\tau(L_u) = L_u + m_{2u}^L + y_{2u}^L, \tau(M_u) = M_u, \tau(Y_{\alpha+v}) = Y_{\alpha+v} + m_{2v}^Y, \quad (7)$$

where $m_{2u}^L, m_{2v}^Y \in M$, $y_{2u}^L \in Y$.

Claim. $\tau = (\varphi_a)^{-1}(\sigma_b^\chi)^{-1}\theta \in \text{Inn}(\mathfrak{gsv}[G, \alpha]) \cdot \Phi$.

In fact, assume

$$\tau(L_0) = L_0 + \sum_{i=1}^p a_{u_i} M_{u_i} + \sum_{j=1}^q b_{v_j} Y_{\alpha+v_j} + b_0 Y_\alpha, \quad \tau(Y_\alpha) = Y_\alpha + \sum_{k=1}^r c_{w_k} M_{w_k}.$$

Applying τ to $[L_0, Y_\alpha] = \alpha Y_\alpha$, we have

$$\sum_{k=1}^r c_{w_k} w_k M_{w_k} - \sum_{j=1}^q b_{v_j} v_j M_{2\alpha+v_j} = \alpha \sum_{k=1}^r c_{w_k} M_{w_k}.$$

Noting that $v_j \neq 0$, $\alpha \neq w_k$, we have

$$q = r, \quad v_k = w_k - 2\alpha, \quad b_{v_k} = \frac{c_{w_k}(w_k - \alpha)}{w_k - 2\alpha}.$$

These gives us

$$\tau(L_0) = L_0 + \sum_{i=1}^p a_{u_i} M_{u_i} + \sum_{k=1}^r \frac{c_{w_k}(w_k - \alpha)}{w_k - 2\alpha} Y_{w_k - \alpha} + b_0 Y_\alpha. \quad (8)$$

Now we construct an inner automorphism τ' for $gsv[G, \alpha]$, which is equal to τ when acting on L_0 and Y_α . Indeed, we set

$$\begin{aligned} \tau' &= \expad\left(\sum_{1 \leq j \neq k \leq r} \frac{c_{w_k} c_{w_j} (\alpha - w_j)(w_j - w_k)}{2(w_k - 2\alpha)(w_j - 2\alpha)(w_j + w_k - 2\alpha)} M_{w_j + w_k - 2\alpha} \right. \\ &\quad \left. + \sum_{k=1}^r \frac{b_0 c_{w_k} (2\alpha - w_k)}{2\alpha w_k} M_{w_k} \right) \\ &\quad \expad\left(-\sum_{i=1}^p \frac{a_{u_i}}{u_i} M_{u_i}\right) \expad\left(\sum_{k=1}^r \frac{c_{w_k}}{2\alpha - w_k} Y_{w_k - \alpha} - \frac{b_0}{\alpha} Y_\alpha\right), \end{aligned}$$

Then one can check that the inner automorphism τ' satisfying

$$\tau'(L_0) = \tau(L_0), \quad \tau'(Y_\alpha) = \tau(Y_\alpha).$$

For any $u \in G$, we apply τ' to $[L_0, L_u] = uL_u$. For the right side, we have

$$u\tau'(L_u) = u\tau(L_u) + u(\tau'(L_u) - \tau(L_u)).$$

For the left side, we have

$$\begin{aligned} [\tau'(L_0), \tau'(L_u)] &= [\tau(L_0), \tau'(L_u)] = [\tau(L_0), \tau(L_u) + (\tau'(L_u) - \tau(L_u))] \\ &= u\tau(L_u) + [\tau(L_0), \tau'(L_u) - \tau(L_u)]. \end{aligned}$$

Thus

$$[\tau(L_0), \tau'(L_u) - \tau(L_u)] = u(\tau'(L_u) - \tau(L_u)). \quad (9)$$

Now we prove the following identity.

$$\tau'(L_u) = \tau(L_u) + e_u M_u. \quad (10)$$

for some $e_u \in \mathbb{F}$.

Indeed, By using (7) and the definition of τ' , one can see that

$$\tau'(M_u) = \tau(M_u) = M_u. \quad (11)$$

So $[\tau'(L_u), M_v] = vM_{u+v}$. Thus $\tau'(L_u) = L_u + m_1 + y_1$ for some $m_1 \in M, y_1 \in Y$. By using (7) again, we have $\tau'(L_u) - \tau(L_u) \in M \oplus Y$. Assuming

$$\tau'(L_u) - \tau(L_u) = \sum_{k=1}^r e_{v_k} M_{v_k} + \sum_{l=1}^s d_{w_l} Y_{\alpha+w_l}.$$

By using this along with identities (8) and (9), we have

$$\begin{aligned} & [L_0 + \sum_{i=1}^p a_{u_i} M_{u_i} + \sum_{j=1}^q b_{v_j} Y_{\alpha+v_j} + b_0 Y_\alpha, \sum_{k=1}^r e_{v_k} M_{v_k} + \sum_{l=1}^s d_{w_l} Y_{\alpha+w_l}] \\ &= \sum_{k=1}^r e_{v_k} v_k M_{v_k} + \sum_{l=1}^s d_{w_l} (\alpha + w_l) Y_{\alpha+w_l} \\ & \quad + \sum_{j=1}^q \sum_{l=1}^s b_{v_j} d_{w_l} (w_l - v_j) M_{2\alpha+w_l+v_j} + \sum_{l=1}^s b_0 d_{w_l} w_l M_{2\alpha+w_l} \\ &= u \left(\sum_{k=1}^r e_{v_k} M_{v_k} + \sum_{l=1}^s d_{w_l} Y_{\alpha+w_l} \right). \end{aligned}$$

By comparing the coefficients of $Y_{\alpha+w_l}$ we have $\sum_{l=1}^s u d_{w_l} = \sum_{l=1}^s d_{w_l} (\alpha + w_l)$. This means $d_{w_l} = 0$ for any $l \in \{1, \dots, s\}$ since $u \neq \alpha + w_l$. Thus the we get

$$\sum_{k=1}^r e_{v_k} v_k M_{v_k} = u \left(\sum_{k=1}^r e_{v_k} M_{v_k} \right).$$

Furthermore, $r = 1, u = v_1$. Thus $\tau'(L_u) - \tau(L_u) = e_u M_u$. This proves (10).

For any $v \in G, v \neq 2\alpha$, by (10), we have

$$\tau'(Y_{\alpha+v}) = (\alpha - \frac{v}{2})^{-1} \tau'[L_v, Y_\alpha] = (\alpha - \frac{v}{2})^{-1} [\tau(L_v) + e_v M_v, \tau(Y_\alpha)] = \tau(Y_{\alpha+v}).$$

For $v = 2\alpha$, we have

$$\tau'(Y_{3\alpha}) = (-3\alpha)^{-1} [\tau'(L_{4\alpha}), \tau'(Y_{-\alpha})] = (-3\alpha)^{-1} [\tau(L_{4\alpha}) + b_{4\alpha} M_{4\alpha}, \tau(Y_{-\alpha})] = \tau(Y_{3\alpha}).$$

In all cases we get

$$\tau'(Y_{\alpha+v}) = \tau(Y_{\alpha+v}), \quad \forall v \in G. \quad (12)$$

Define

$$\phi : \mathfrak{gsv}[G, \alpha] \rightarrow \mathfrak{gsv}[G, \alpha] : L_u \mapsto L_u + e_u M_u, \quad M_u \mapsto M_u, \quad Y_{\alpha+v} \mapsto Y_{\alpha+v}.$$

By (10), (11) and (12), we have $\tau' = \tau\phi$. Thus $\tau = \tau'\phi^{-1} \in \text{Inn}(\mathfrak{gsv}[G, \alpha]) \cdot \Phi$. Which completes the proof of the Claim.

By the claim, we have

$$\theta = \sigma_b^x \varphi_a \tau' \phi^{-1} \in (Hom(T, \mathbb{F}^*) \times \mathbb{F}^*) \cdot S(G, T) \cdot Inn(\mathfrak{gsv}[G, \alpha]) \cdot \Phi.$$

Since $Inn(\mathfrak{gsv}[G, \alpha])$ is a normal subgroup of $Aut(\mathfrak{gsv}[G, \alpha])$, thus $Inn(\mathfrak{gsv}[G, \alpha]) \cdot \Phi = \Phi \cdot Inn(\mathfrak{gsv}[G, \alpha])$. One can check straightforward that the following two facts hold:

$$(Hom(T, \mathbb{F}^*) \times \mathbb{F}^*) \triangleleft (Hom(T, \mathbb{F}^*) \times \mathbb{F}^*) \cdot S(G, T),$$

and

$$\Phi \triangleleft (Hom(T, \mathbb{F}^*) \times \mathbb{F}^*) \cdot S(G, T) \cdot \Phi.$$

Thus

$$Aut(\mathfrak{gsv}[G, \alpha]) \simeq (((Hom(T, \mathbb{F}^*) \times \mathbb{F}^*) \rtimes S(G, T)) \ltimes \Phi) \ltimes Inn(\mathfrak{gsv}[G, \alpha]).$$

This completes the proof of the Theorem 3.2. \square

4. Verma modules of $\mathfrak{gsv}[G, \alpha]$

In this section we construct and investigate the structure of Verma modules over the generalized Schrödinger-Virasoro algebra $\mathfrak{gsv}[G, \alpha]$.

Note that $T = G \cup G_1$ is a subgroup of \mathbb{F} , we fix a total order " \succeq " on T which is compatible with the addition, i.e., $x \succeq y$ implies $x + z \succeq y + z$ for any $z \in T$ (see [4],[9]). We write $x \succ y$ if $x \succeq y$ and $x \neq y$. Let

$$T_+ := \{x \in T | x \succ 0\}, \quad T_- := \{x \in T | x \prec 0\}.$$

Then $T = T_+ \cup \{0\} \cup T_-$ and $\mathfrak{gsv}[G, \alpha]$ has a triangular decomposition:

$$\mathfrak{gsv}[G, \alpha] = \mathfrak{gsv}[G, \alpha]_- \oplus \mathfrak{gsv}[G, \alpha]_0 \oplus \mathfrak{gsv}[G, \alpha]_+,$$

where

$$\begin{aligned} \mathfrak{gsv}[G, \alpha]_- &= \bigoplus_{u \prec 0} \mathbb{F}L_u \oplus \bigoplus_{u \prec 0} \mathbb{F}M_u \oplus \bigoplus_{\alpha+v \prec 0} \mathbb{F}Y_{\alpha+v}, \\ \mathfrak{gsv}[G, \alpha]_+ &= \bigoplus_{u \succ 0} \mathbb{F}L_u \oplus \bigoplus_{u \succ 0} \mathbb{F}M_u \oplus \bigoplus_{\alpha+v \succ 0} \mathbb{F}Y_{\alpha+v} \end{aligned}$$

and $\mathfrak{gsv}[G, \alpha]_0 = \mathbb{F}L_0 \oplus \mathbb{F}M_0$. The universal enveloping algebra of $\mathfrak{gsv}[G, \alpha]$ is given by

$$U(\mathfrak{gsv}[G, \alpha]) = U(\mathfrak{gsv}[G, \alpha])_- U(\mathfrak{gsv}[G, \alpha])_0 U(\mathfrak{gsv}[G, \alpha])_+.$$

The elements $L_{i_1} \cdots L_{i_r} M_{j_1} \cdots M_{j_s} Y_{\alpha+k_1} \cdots Y_{\alpha+k_t}$, where $r, s, t \in \mathbb{N}, i_1 \succeq \cdots \succeq i_r, j_1 \succeq \cdots \succeq j_s, k_1 \succeq \cdots \succeq k_t$, along with 1, form a basis of $U(\mathfrak{gsv}[G, \alpha])$.

Let $c, h \in \mathbb{F}$, V_h be a 1-dimensional vector space over \mathbb{F} spanned by v_h , i.e., $V_h = \mathbb{F}v_h$. View V_h as a $\mathfrak{gsv}[G, \alpha]_0$ -module such that $L_0.v_h = hv_h$, $M_0.v_h = cv_h$. Then V_h is a $\mathfrak{B} = \mathfrak{gsv}[G, \alpha]_+ \oplus \mathfrak{gsv}[G, \alpha]_0$ -module by setting $\mathfrak{gsv}[G, \alpha]_+.V_h = 0$.

Definition 4.1. The induced module $V(c, h) = \text{Ind}_{\mathfrak{g}}^{\mathfrak{gsv}[G, \alpha]} V_h = U(\mathfrak{gsv}[G, \alpha]) \otimes_{U(\mathfrak{g})} V_h$ is called the Verma module of $\mathfrak{gsv}[G, \alpha]$ with highest weight (c, h) .

Let $U := U(\mathfrak{gsv}[G, \alpha])$. For any $c, h \in \mathbb{F}$, let $I(c, h)$ be the left ideal of U generated by the elements

$$\{L_u, M_u, Y_{\alpha+v} | u \in G_+, \alpha + v \in G_{1+}\} \cup \{L_0 - h, M_0 - c\},$$

where $G_+ = G \cap T_+$, $G_{1+} = G_1 \cap T_+$. Then the Verma module with highest weight (c, h) for $\mathfrak{gsv}[G, \alpha]$ also can be defined as $V(c, h) := U/I(c, h)$.

By definition, we can easily get a basis of $V(c, h)$ consisting of all vectors of the form:

$$v_h, L_{-i_1} \cdots L_{-i_r} M_{-j_1} \cdots M_{-j_s} Y_{\alpha-k_1} \cdots Y_{\alpha-k_t} v_h,$$

where

$$0 \prec i_1 \preceq \cdots \preceq i_r, 0 \prec j_1 \preceq \cdots \preceq j_s, \alpha \prec k_1 \preceq \cdots \preceq k_t; r, s, t \in \mathbb{N}.$$

Remark. One can see that M_0 acts as a scalar c on $V(c, h)$ since $\mathbb{F}M_0$ is the center of $\mathfrak{gsv}[G, \alpha]$. Next, we call a vector $v \in V(c, h)$ a weight vector with weight μ means v satisfying $L_0 v = \mu v$.

Lemma 4.2. $V(c, h)$ is a weight module of $\mathfrak{gsv}[G, \alpha]$, and $V(c, h) = \bigoplus_{\mu \in h - T_+} V_\mu$, where $V_\mu = \text{span}\{L_{-i_1} \cdots L_{-i_r} M_{-j_1} \cdots M_{-j_s} Y_{\alpha-k_1} \cdots Y_{\alpha-k_t} v_h \mid \sum_{p=1}^r i_p - \sum_{p=1}^s j_p - \sum_{p=1}^t (\alpha - k_p) = h - \mu\}$ is the weight vector space with weight μ .

Proof. It suffices to show that L_0 acts diagonally on the basis elements of $V(c, h)$. By the definition of v_h , $L_0 v_h = h v_h$. Suppose $u \in V(c, h)$ such that $L_0 u = a u$. Then

$$L_0(L_{-i} u) = (a - i) L_{-i} u, L_0(M_{-j} u) = (a - j) M_{-j} u, L_0(Y_{\alpha-k} u) = (a + \alpha - k) Y_{\alpha-k} u.$$

Thus Lemma 4.2 holds. \square

We know from [4] that for the fixed total order " \succeq " of T , either " \succeq " is dense, i.e., $\forall x \in T_+$, the cardinality of $\{y \in T \mid 0 \prec y \prec x\}$ is infinite, or " \succeq " is discrete, i.e., there exists $a \in T$ such that the set $\{y \in T \mid 0 \prec y \prec a\}$ is empty.

For the generalized Virasoro algebra $Vir[G]$ studied in [4], the irreducibility of Verma module over $Vir[G]$ is depends on whether the total order of G is dense or discrete (see Theorem 3.1 in [4]). With respect to Verma modules over generalized Witt algebras studied in [8], the irreducibility depends on the action of L_0 on the highest weight vector (see Theorem 3 in [9]). It is very interesting that the irreducibility of Verma modules over $\mathfrak{gsv}[G, \alpha]$ depends on neither the action of L_0 nor whether the total order is dense or discrete, we point out that the irreducibility just depends on the action of the element M_0 .

For $x \in V(c, h)$, we set

$$x = \sum_{\substack{i_1 \preceq \dots \preceq i_r, j_1 \preceq \dots \preceq j_s, k_1 \preceq \dots \preceq k_t \\ i_1, \dots, i_r, j_1, \dots, j_s \in G_+, k_1 - \alpha, \dots, k_t - \alpha \in G_{1+}}} a_{\underline{i}, \underline{j}, \underline{k}} L_{-i_1} \cdots L_{-i_r} M_{-j_1} \cdots M_{-j_s} Y_{\alpha - k_1} \cdots Y_{\alpha - k_t} v_h.$$

where $a_{\underline{i}, \underline{j}, \underline{k}} \in \mathbb{F}$, $\underline{i} = (i_1, \dots, i_r)$, $\underline{j} = (j_1, \dots, j_s)$, $\underline{k} = (\alpha - k_1, \dots, \alpha - k_t)$, and only finitely many $a_{\underline{i}, \underline{j}, \underline{k}} \neq 0$. We define

$$A_x := \{\underline{i} = (i_1, \dots, i_r) | a_{\underline{i}, \underline{j}, \underline{k}} \neq 0 \text{ for some } \underline{j}, \underline{k}\}, l = \max\{r | \underline{i} = (i_1, \dots, i_r) \in A_x\},$$

where $l = 0$ if $A_x = \emptyset$. We also define l to be the length of the element x , and denote it by $\text{len}(x)$, i.e., $l = \text{len}(x)$.

For $r \in \mathbb{N}$, we set

$$V_r := \text{span}_{\mathbb{F}}\{x | \text{len}(x) \leq r\}.$$

In what follows, we assume $V_r = 0$ if $r \leq -1$. One can check the following two lemmas by straightforward and easy computations.

Lemma 4.3. (i)

$$\begin{aligned} & M_j L_{-i_1} L_{-i_2} \cdots L_{-i_r} M_{-j_1} \cdots M_{-j_s} Y_{\alpha - k_1} \cdots Y_{\alpha - k_t} v_h \\ \equiv & -j \left(\sum_{1 \leq p \leq r} L_{-i_1} \cdots \hat{L}_{-i_p} \cdots L_{-i_r} M_{j - i_p} \right) M_{-j_1} \cdots M_{-j_s} Y_{\alpha - k_1} \cdots Y_{\alpha - k_t} v_h \pmod{V_{r-2}}, \end{aligned}$$

for any $r \in \mathbb{N}; j \in G_+; 0 \prec i_1 \preceq \dots \preceq i_r; 0 \prec j_1 \preceq \dots \preceq j_s; 0 \prec k_1 - \alpha \preceq \dots \preceq k_t - \alpha$, where $\hat{}$ means the corresponding element is deleted.

(ii)

$$\begin{aligned} & L_j M_{-i_1} M_{-i_2} \cdots M_{-i_r} v_h \\ = & \left(\sum_{1 \leq p \leq r} (-i_p) M_{-i_1} \cdots \hat{M}_{-i_p} \cdots M_{-i_r} M_{j - i_p} \right) v_h, \end{aligned}$$

for any $r \in \mathbb{N}; j, i_1, \dots, i_r \in G_+$. In particular,

$$L_j M_{-i_1} M_{-i_2} \cdots M_{-i_r} v_h = 0, \forall j \succ \max\{i_1, \dots, i_r\}.$$

Lemma 4.4. (i) $Y_{-\alpha+j} Y_{\alpha-k_1} Y_{\alpha-k_2} \cdots Y_{\alpha-k_t} v_h = 0$, $\forall j \succ k_t$, where $\alpha \prec k_1 \preceq \dots \preceq k_t$.

(ii) If $M_0.v_h = 0$, then $Y_{-\alpha+j} Y_{\alpha-k_1} Y_{\alpha-k_2} \cdots Y_{\alpha-k_t} v_h = 0$, $\forall j \succeq k_t$, where $\alpha \prec k_1 \preceq \dots \preceq k_t$.

Corollary 4.5. $M_j V_r \subseteq V_{r-1}$, for any $j \in G_+$.

Proof. It follows Lemma 4.3 (i) immediately. \square

Theorem 4.6. (i) The Verma module $V(c, h)$ is an irreducible $\mathfrak{gs}\mathfrak{v}[G, \alpha]$ module if $c \neq 0$.

(ii) If $c = 0$, then the Verma module $V(0, h)$ contains a unique maximal proper submodule $N(0, h)$, where $N(0, h)$ is generated by $\{L_{-u}v_h, M_{-u}v_h, Y_{\alpha-v}v_h | u \in G_+, v - \alpha \in G_{1+}\}$ if $h = 0$, by $\{M_{-u}v_h, Y_{\alpha-v}v_h | u \in G_+, v - \alpha \in G_{1+}\}$ if $h \neq 0$.

Proof. (i) Suppose $c \neq 0$. Let $u_0 \neq 0$ be any given weight vector in $V(c, h)$. By Lemma 4.2 and the fact that a submodule of a weight module is a weight module, we need only to prove that $v_h \in U(\mathfrak{gsv}[G, \alpha])u_0$.

Claim I. There exists a weight vector $u \in U(\mathfrak{gsv}[G, \alpha])u_0$ such that

$$u = \sum_{\substack{j_1 \preceq \dots \preceq j_s; \ k_1 \preceq \dots \preceq k_t \\ j_1, \dots, j_s \in G_+; k_1 - \alpha, \dots, k_t - \alpha \in G_{1+}}} a_{\underline{j}, \underline{k}} M_{-j_1} \cdots M_{-j_s} Y_{-k_1 + \alpha} \cdots Y_{-k_t + \alpha} v_h,$$

where $a_{\underline{j}, \underline{k}} \in \mathbb{F}$ and only finitely many $a_{\underline{j}, \underline{k}} \neq 0$, $\underline{j} = (j_1, \dots, j_s)$, $\underline{k} = (k_1, \dots, k_t)$.

In fact, suppose

$$\begin{aligned} u_0 &= \sum_{\substack{0 \prec i_1 \preceq \dots \preceq i_r, \\ 0 \prec j_1 \preceq \dots \preceq j_s, \\ \alpha \prec k_1 \preceq \dots \preceq k_t.}} a_{\underline{i}, \underline{j}, \underline{k}} L_{-i_1} \cdots L_{-i_r} M_{-j_1} \cdots M_{-j_s} Y_{\alpha - k_1} \cdots Y_{\alpha - k_t} v_h \\ &\equiv \sum_{\substack{0 \prec i_1 \preceq \dots \preceq i_l, \\ 0 \prec j_1 \preceq \dots \preceq j_s, \\ \alpha \prec k_1 \preceq \dots \preceq k_t.}} a_{\underline{i}, \underline{j}, \underline{k}} L_{-i_1} \cdots L_{-i_l} M_{-j_1} \cdots M_{-j_s} Y_{\alpha - k_1} \cdots Y_{\alpha - k_t} v_h \pmod{V_{l-1}}, \end{aligned}$$

where $l = \text{len}(u_0)$, $\underline{i} = (i_1, \dots, i_r)$, $\underline{j}, \underline{k}$ as above, $a_{\underline{i}, \underline{j}, \underline{k}} \in \mathbb{F}$.

If $l = 0$, there is nothing to prove. Now suppose $\text{len}(u_0) = l \geq 1$, we denote

$$i_l^{(0)} := \max\{i_l | (i_1, \dots, i_l) \in A_{u_0}\},$$

where $A_{u_0} := \{\underline{i} | \underline{i} = (i_1, \dots, i_l), a_{\underline{i}, \underline{j}, \underline{k}} \neq 0 \text{ for some } \underline{j}, \underline{k}\}$, then by using Lemma 4.3 (i) and Corollary 4.4 we can deduce that

$$\begin{aligned} u_1 &= M_{i_l^{(0)}} u_0 = \sum a_{\underline{i}, \underline{j}, \underline{k}} M_{i_l^{(0)}} L_{-i_1} \cdots L_{-i_r} M_{-j_1} \cdots M_{-j_s} Y_{\alpha - k_1} \cdots Y_{\alpha - k_t} v_h \\ &\equiv \sum a_{\underline{i}^{(1)}, \underline{j}^{(1)}, \underline{k}^{(1)}} L_{-i_1^{(1)}} \cdots L_{-i_{l-1}^{(1)}} M_{-j_1^{(1)}} \cdots M_{-j_s^{(1)}} Y_{\alpha - k_1^{(1)}} \cdots Y_{\alpha - k_t^{(1)}} v_h \pmod{V_{l-2}}. \end{aligned}$$

it is clear that $u_1 \neq 0$ and $\text{len}(u_1) = l - 1$.

Repeating the process and define u_s recursively for $s = 2, \dots, l$, one obtains the claim.

Claim II. There exists an weight vector $w \in U(\mathfrak{gsv}[G, \alpha])u_0$ such that w takes the following form

$$w = \sum_{s \geq 0, j_1 \preceq \dots \preceq j_s; j_1, \dots, j_s \in G_+} a_{\underline{j}} M_{-j_1} \cdots M_{-j_s} v_h.$$

In fact, by Claim I, we know that there is a weight vector $u \in U(\mathfrak{gsv}[G, \alpha])u_0$ such that

$$u = \sum_{\substack{s, t \geq 0, j_1 \preceq \dots \preceq j_s \\ k_1 \preceq \dots \preceq k_t; j_1, \dots, j_s \in G_+ \\ k_1 - \alpha, \dots, k_t - \alpha \in G_{1+}}} a_{\underline{j}, \underline{k}} M_{-j_1} \cdots M_{-j_s} Y_{-k_1 + \alpha} \cdots Y_{-k_t + \alpha} v_h.$$

Set

$$B := \{\underline{k} = (k_1, k_2, \dots, k_t) | a_{\underline{j}, \underline{k}} \neq 0 \text{ for some } \underline{j}\}, k^{(0)} = \max\{k_t | \underline{k} \in B\}.$$

Then by Lemma 4.4, we have

$$w_1 = Y_{-(\alpha - k^{(0)})} u = \sum a_{\underline{j}, \underline{k}^{(1)}} M_{-j_1} \cdots M_{-j_s} Y_{\alpha - k_1^{(1)}} \cdots Y_{\alpha - k_t^{(1)}} (M_0 v_h).$$

Noting that $w_1 \neq 0$ and $w_1 \in U(\mathfrak{gsv}[G, \alpha])u_0$ is a weight vector. One repeats the process to get Claim II.

From Claim II we know that there exists a weight vector $w \in U(\mathfrak{gsv}[G, \alpha])u_0$ such that it has the following form

$$w = \sum_{s \geq 0, 0 \prec j_1 \preceq \dots \preceq j_s} a_{\underline{j}} M_{-j_1} \cdots M_{-j_s} v_h.$$

We define

$$\text{length}(w) = \max\{s | \underline{j} = (j_1, \dots, j_s), a_{\underline{j}} \neq 0\}.$$

If $\text{length}(w) = 0$, then $v_h \in U(\mathfrak{gsv}[G, \alpha])u_0$ and (i) holds. Now suppose $\text{length}(w) > 0$. Denote $j^{(0)} = \max\{j_s | \underline{j} = (j_1, \dots, j_s), a_{\underline{j}} \neq 0\}$. By applying $L_{j^{(0)}}$ to w and using Lemma 4.3 (ii), we have

$$0 \neq w_1 = L_{j^{(0)}} w = \sum_{j_1^{(1)} \preceq \dots \preceq j_s^{(1)}; j_1^{(1)}, \dots, j_s^{(1)} \in G_+} a_{\underline{j}}^{(1)} M_{-j_1^{(1)}} \cdots M_{-j_s^{(1)}} M_0 v_h.$$

It is clear that

$$\text{length}(w_1) < \text{length}(w).$$

Repeating the process, we obtain

$$0 \neq w_s = a M_0 v_h = a c v_h \in U(\mathfrak{gsv}[G, \alpha])u_0$$

for some $0 \neq a \in \mathbb{F}$. So $v_h \in U(\mathfrak{gsv}[G, \alpha])u_0$ and $V(c, h)$ is irreducible.

(ii) If $c = 0, h = 0$, by the definition of $N(0, 0)$, one knows that all the basis elements of $V(0, 0)$ except v_h are clearly in $N(0, 0)$. It suffices to show that $v_h \notin N(0, 0)$. For any weight vector $v \in N(0, 0)$, suppose the weight of v is μ , and

for any basis element $L_{i_1} \cdots L_{i_r} M_{j_1} \cdots M_{j_s} Y_{\alpha+k_1} \cdots Y_{\alpha+k_t}$ of $U(\mathfrak{gs}\mathfrak{v}[G, \alpha])$ such that $\sum_{p=1}^r i_p + \sum_{p=1}^s j_p + \sum_{p=1}^t (\alpha + k_p) = -\mu$, we have

$$L_{i_1} \cdots L_{i_r} M_{j_1} \cdots M_{j_s} Y_{\alpha+k_1} \cdots Y_{\alpha+k_t} v = aL_0 v_h + bM_0 v_h = 0,$$

for some $a, b \in \mathbb{F}$. This implies that $v_h \notin N(0, 0)$.

If $c = 0, h \neq 0$, similarly as above, we can see that $U(L_-) \not\subset N(0, h)$, where $L_- = \bigoplus_{u < 0} \mathbb{F} L_u$. This means that $N(0, h)$ is a proper submodule of $V(0, h)$. Suppose V is any submodule of $V(0, h)$ such that $V \supsetneq N(0, h)$, then there exist $i_1, \dots, i_r \in G_+, r \in \mathbb{N}$ such that $L_{-i_1} \cdots L_{-i_r} v_h \in V$. If $r = 0$, then $v_h \in V$ and $V = V(0, h)$. Suppose $r \geq 1$. We denote $i = i_1 + i_2 + \cdots + i_r$, then

$$L_i L_{-i_1} \cdots L_{-i_r} v_h = (-1)^r (i + i_1)(i - i_1 + i_2) \cdots (i - i_1 - i_2 - \cdots - i_{r-1} + i_r) h v_h \in V.$$

Since $(-1)^r (i + i_1)(i - i_1 + i_2) \cdots (i - i_1 - i_2 - \cdots - i_{r-1} + i_r) h \neq 0$ we have $v_h \in V$ and $V = V(0, h)$. So $N(0, h)$ is the unique maximal proper submodule of $V(0, h)$. \square

Remark. $V(0, 0)/N(0, 0) \simeq \mathbb{F}$ is a trivial module of $\mathfrak{gs}\mathfrak{v}[G, \alpha]$. $V(0, h)/N(0, h) \simeq U(L_-)$ as vector space.

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